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ON MACROSCOPIC SPATIAL AVERAGING OF CONTACT AND LONG-RANGE MICROSCOPIC FORCES

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Dedicated to Dr R. Zahradník on the occasion of his 60th birthday.

Even though ensemble averages are sometimes equivalent, or even preferable, to spatial averages, the latter have incontestable significance. Spatial averaging, which has been restricted until now to basic electromagnetic equations, is applied here to the Cauchy equation of movement. Spatial filtration of microscopic properties is especially useful in the case of continuous media subjected to long-range forces, for which no intuitive macroscopic reasoning is possible.

In a continuous medium, long-range microscopic forces also reach the particles (atoms, electrons, etc.) of the inner part of the medium. However, the volume distribution of these microscopic forces may result, through macroscopic averaging, in a surface force distribution exerted on the face of the body. For example, this is the case for a dielectric continuous medium which is electrically polarized. As a consequence of this, in macroscopic equations, there is not necessarily a canonical (or intrinsic) separation between long-range forces and surface contact pressure forces. This explains why the simple addition of a term representing the long-range forces is not sufficient to extend the validity of macroscopic equations to cases in which a medium is submitted to long-range forces.

A good example of an intuitive reasoning which is difficult to generalize in the presence of long-range forces is the derivation of the Cauchy equation of movement[1] of a medium,

$$\nabla . \boldsymbol{T} + \boldsymbol{f} - \varrho \boldsymbol{\Gamma} = \mathbf{o} , \qquad (1)$$

which is a basic equation in chemical physics and chemical engineering. In this article, we shall concentrate our attention on this equation which links the pressure or stress tensor $\boldsymbol{T}$ to the distribution of external forces $\boldsymbol{f}$ exerted on the medium, and to the acceleration $\boldsymbol{\Gamma}$ of the medium; ϱ is the mass density.

$\boldsymbol{T}$, $\boldsymbol{f}$, ϱ, $\boldsymbol{\Gamma}$ are macroscopic parameters, which can be defined either as ensemble averages *for a given statistical state,*

$$\langle G \rangle_{\text{ens}} = \int \Phi^{(N)} G \, \mathrm{d}v ,$$

using the distribution function $\Phi^{(N)}$ of the statistical state for medium N, or alternatively, as spatial averages *for a given instantaneous configuration of the bodies.* In the case of ensemble averages, the Cauchy equation can be derived exactly by using the Liouville theorem twice (see for example ref.[2]). In the case of spatial averages, Eq. (*1*) remains to be proved and its interpretation clarified, which is our intention in this paper.

THEORETICAL

Like many equations dealing with continuous media, Eq. (*1*) is obtained by considering a macroscopic elementary volume of the medium, which is a way of introducing macroscopic spatial averages of microscopic parameters. In the case where medium N exerts no long-range forces on itself, Eq. (*1*) can be established by applying the principle of dynamics $\mathbf{F} = m\boldsymbol{\Gamma}$ to a unit elementary volume B of the considered medium N. To avoid ambiguities in the definition of bodies, and hence of forces, let us adopt the notation $X \setminus Y$ for the part of body X which is not included in body Y; X will be omitted when it stands for all the existing bodies, so that $\setminus Y$ represents the outside of Y. The resultant force $\mathbf{F}(\setminus B \rightarrow B)$ exerted on B by the outside is the sum of (i) the force distribution $\mathbf{f} \equiv \mathbf{f}(\setminus N \rightarrow B)$ exerted on B by the outside of the medium and (ii) the surface force distribution

$$\mathbf{f}(N \setminus B \rightarrow B) = \mathbf{n} \cdot \mathbf{T} \tag{2}$$

exerted on the surface of B by the rest of the medium ($\mathbf{n}$ is the outer unit vector normal to the surface of B). We have

$$\iint_B \mathrm{d}s\, \mathbf{f}(N \setminus B \rightarrow B) = \int_B \mathrm{d}v\, \nabla \cdot \mathbf{T}\,, \tag{3}$$

and Eq. (*1*) follows. In the case where the medium $N \setminus B$ also exerts long-range forces on B, a macroscopic force distribution $\mathbf{f}'(N \setminus B \rightarrow B)$ must be added to the previous surface force distribution $\mathbf{f}(N \setminus B \rightarrow B)$. It is then difficult to distinguish between these two distributions because $\mathbf{f}'$ may comprise a surface distribution. A priori, relation (*2*) between the surface force $\mathbf{f}(N \setminus B \rightarrow B)$ and $\mathbf{n} \cdot \mathbf{T}$ is no longer valid. In addition, it is not clear whether the force distribution $\mathbf{f}'(N \setminus B \rightarrow B)$ can be added to $\mathbf{f}$ in Eq. (*1*), because the definition of $\mathbf{T}$ already takes into account the microscopic interactions corresponding to this force. For these two reasons, Eq. (*1*) ceases to be intuitive.

We shall show now that Eq. (*1*) is valid in spatial average in all cases. For proving this we shall use a more accurate manner of performing the averaging operation which consists of spatial filtration[2,3]. This process has been well illustrated in the macroscopic equations for electromagnetism[3]. The spatial average $\langle G \rangle(\mathbf{x})$ of a physical parameter $G(\mathbf{x})$ defined in space (G is a function or a distribution) is obtained[2,3] by convolution of G with a spatial filter function:

$$\langle G \rangle = G * a$$
$$\langle G(\mathbf{x}) \rangle = \int G(\mathbf{u})\, a(\mathbf{x} - \mathbf{u})\, \mathrm{d}^3\mathbf{u}\,. \tag{4}$$

The filter function a is chosen once and for all. Its support or carrier is concentrated in a neighborhood of the origin around which the function has the spherical symmetry: $a(\mathbf{x}) = a(r)$; a is normalized to unity: $\int a(\mathbf{x})\, \mathrm{d}v = 1$. Either a bell or a crenel--shaped profile may be chosen for a.

The radius R beyond which a is almost negligible (i.e. $\int_0^R 4\pi r^2\, a(r)\, \mathrm{d}r \simeq 1$) defines the macroscopic scale.

Macroscopic Force

In spatial averages, the macroscopic force exerted on a body Y by a body X, at a given time, is defined as

$$\langle \mathbf{f}(X \to Y) \rangle = \sum_{\substack{i \in X \\ i \neq j}} \sum_{j \in Y} \mathbf{f}(i \to j)\, \delta_j * a\,. \tag{5}$$

The condition $i \neq j$ concerning the particles i, j, is irrelevant when the bodies X and Y are disjoint. The internal macroscopic force

$$\langle \mathbf{f}(X \to X) \rangle = \Big(\sum_{i \in X} \mathbf{f}(X \setminus i \to i)\, \delta_i\Big) * a \tag{6}$$

is not zero *a priori*, even for forces which satisfy the action and reaction equality principle. In fact,

$$\langle \mathbf{f}(X \to X) \rangle = \sum_{\substack{i \in X \\ i < j}} \sum_{j \in X} \mathbf{f}(i \to j)\, (\delta_j - \delta_i) * a \tag{7}$$

is not zero in general, because the distances from particles i and j to the observation point $\mathbf{x}$ differ in general; it is different from zero even when the filter function a is chosen to be uniform, i.e. of uniform value of $(\frac{4}{3}\pi R^3)^{-1}$ in the ball $(\mathbf{O}, R)$. In terms of distributions, it is therefore important to distinguish between the external force $\langle \mathbf{f}(\setminus X \to X) \rangle$ and the macroscopic total force

$$\langle \mathbf{f}(\to X) \rangle = \langle \mathbf{f}(\setminus X \to X) \rangle + \langle \mathbf{f}(X \to X) \rangle \tag{8}$$

exerted by all the existing bodies.

Divergence of the Pressure Tensor

Let A be any body. The pressure tensor of A is the sum of the particles pressure tensor $\boldsymbol{T}_p$ and of the field pressure tensor $\boldsymbol{T}_f$ of A. In spatial averages, the mean velocity of the body A at point $\boldsymbol{x}$ is

$$\boldsymbol{V}(\boldsymbol{x}) = \frac{\langle \boldsymbol{p} \rangle}{\langle \varrho \rangle} \equiv \left(\sum_{i \in A} m_i \delta_i * a \right)^{-1} \sum_{i \in A} m_i \boldsymbol{v}_i \delta_i * a \,. \tag{9}$$

The microscopic pressure tensor of the particles of body A is defined as the distribution:

$$\boldsymbol{T}_b = - \sum_{i \in A} m_i \boldsymbol{v}_i^2 \delta_i \,, \tag{10}$$

(here the tensor product is written without a multiplication sign), or rather

$$\boldsymbol{T}_p = - \sum_{i \in A} m_i (\boldsymbol{v}_i - \boldsymbol{V})^2 \, \delta_i \,. \tag{11}$$

Its spatial average is

$$\langle \boldsymbol{T}_p \rangle = \boldsymbol{T}_p * a = - \sum_{i \in A} m_i \boldsymbol{v}_i^2 \, a(\boldsymbol{x} - \boldsymbol{i}) + \langle \varrho \rangle \, \boldsymbol{V}^2 \,. \tag{12}$$

In ensemble average, the acceleration is defined as

$$\boldsymbol{\Gamma}_1 = \frac{\partial \boldsymbol{V}_1}{\partial t} + \boldsymbol{V}_1 \,.\, \nabla \boldsymbol{V}_1 \,, \tag{13}$$

by means of the average velocity $\boldsymbol{V}_1 \equiv \boldsymbol{V}_1(\boldsymbol{x}, t) \equiv \langle \boldsymbol{p} \rangle_{\text{ens}} / \langle \varrho \rangle_{\text{ens}}$. In spatial averages, it can be defined in a similar way, using the time derivative $\mathrm{d}\boldsymbol{V}/\mathrm{d}t$ of $\boldsymbol{V}(\boldsymbol{x}; (\boldsymbol{i}), (\boldsymbol{p}_i), t)$ (where $\boldsymbol{i}$ and $\boldsymbol{p}_i$ represent the microscopic positions and momenta) considered at a fixed point $\boldsymbol{x}$:

$$\boldsymbol{\Gamma}(\boldsymbol{x}) = \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} + \boldsymbol{V} \,.\, \nabla \boldsymbol{V} \,. \tag{14}$$

Obviously, we have $\boldsymbol{\Gamma} \neq \boldsymbol{\Gamma}_1$, as $\boldsymbol{V} \neq \boldsymbol{V}_1$.

The time derivative, which transforms a distribution $G((\boldsymbol{i}), (\boldsymbol{p}_i), t)$ into

$$\frac{\mathrm{d}G}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \, G((\boldsymbol{i}(t)), (\boldsymbol{p}_i(t)), t) \,, \tag{15}$$

can be applied either to a microscopic distribution G or to its spatial average $\langle G \rangle$. The $\mathrm{d}/\mathrm{d}t$ notation is assumed because the derivative operates on all occurences of t in G, i.e. is a total derivative. The $\partial/\partial t$ notation, as in Eq. (*13*), is possible too for the same derivative (*15*), to emphasize the fact that the derivative does not operate on the spatial variable $\boldsymbol{x}$ of distribution G. The $\mathrm{d}/\mathrm{d}t$ derivative commutes with the convolution $*$. This property of the mathematical distributions leads to the equalities

$$\left\langle \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \right\rangle = \frac{\mathrm{d}\langle \boldsymbol{p} \rangle}{\mathrm{d}t} \tag{16}$$

and

$$\frac{\mathrm{d}\langle \varrho \rangle}{\mathrm{d}t} \boldsymbol{V} = -\nabla . (\langle \varrho \rangle \boldsymbol{V}) \boldsymbol{V} , \tag{17}$$

which are not trivial from a physical point of view, because of the various physical terms composing the derivatives; the equations corresponding to Eqs (*16*) and (*17*) in ensemble average are consequences of the Liouville theorem. Using Eqs (*12*), (*14*), (*16*), (*17*), we obtain

$$\nabla . \langle \boldsymbol{T}_p \rangle = - \sum_{i \in A} m_i \boldsymbol{v}_i^2 . \nabla a(\boldsymbol{x} - \boldsymbol{i}) + \nabla \langle \varrho \rangle . \boldsymbol{V}^2 + \langle \varrho \rangle \boldsymbol{V} . \nabla \boldsymbol{V} + \langle \varrho \rangle (\nabla . \boldsymbol{V}) \boldsymbol{V} =$$

$$= \langle \varrho \rangle \boldsymbol{\Gamma} - \sum_{i \in A} m_i \gamma_i \delta_i * a , \tag{18}$$

where γ_i is the acceleration of particle i. The microscopic equation of movement allows the replacement of $m_i \gamma_i$ by the total microscopic force

$$\boldsymbol{f}(\setminus i \to i) = \boldsymbol{f}(\setminus A \to i) + \boldsymbol{f}(A \setminus i \to i) \tag{19}$$

exerted on particle i. Therefore we obtain

$$\nabla . \langle \boldsymbol{T}_p \rangle = \langle \varrho \rangle \boldsymbol{\Gamma} - \langle \boldsymbol{f}(\setminus A \to A) \rangle = \langle \boldsymbol{f}(A \to A) \rangle . \tag{20}$$

The microscopic pressure tensor $\boldsymbol{T}_f$ of the electromagnetic field is defined[4] as the distribution

$$\boldsymbol{T}_f = \varepsilon_0 \boldsymbol{E}^2 + \mu_0^{-1} \boldsymbol{B}^2 = \tfrac{1}{2}(\varepsilon_0 E^2 + \mu_0^{-1} B^2) \boldsymbol{I}_2 , \tag{21}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the microscopic electromagnetic fields created by A, and $\boldsymbol{I}_2 \equiv \equiv \mathrm{diag}(1, 1, 1)$ is the identity tensor of order two. Tensor (*21*) has been primarily devised to represent the r^{-1} interactions, but Eq. (*21*) can also take into account short range interactions[5,6] (in r^{-6}, r^{-12}, etc.), since these interactions are actually weighted averages of r_{ij}^{-1} interactions. Other potential pressure tensors, specially devised[7] for various potential shapes, are often used, allowing a reduction in the number of particles. It is possible to consider here only tensor (*21*) because other pressure tensors are only approximations of Eq. (*21*), as the potentials r^{-6}, r^{-12}, etc. are only approximations of r^{-1} interacting particle densities.

Gravitational forces are generally dealt with separately, and therefore the Earth is excluded here from the assumed body A. The gravitational field tensor should be added to expression (*21*), but it is negligible in this case.

The divergence of $\boldsymbol{T}_f$ is[4]

$$\nabla \cdot \boldsymbol{T}_f = \sum_{i \in A} \sum_{\substack{j \in A \\ i \neq j}} \left[\mathbf{f}(i \to j)\, \delta_j + \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E}_i \wedge \mathbf{B}_j) \right], \tag{22}$$

and the divergence of the spatial average $\langle \boldsymbol{T}_f \rangle = \boldsymbol{T}_f * a$ is therefore

$$\nabla \cdot \langle \boldsymbol{T}_f \rangle = \langle \mathbf{f}(A \to A) \rangle + \left(\varepsilon_0 \frac{\partial}{\partial t} \sum_{i \in A} \sum_{\substack{j \in A \\ i \neq j}} \mathbf{E}_i \wedge \mathbf{B}_j \right) * a\,. \tag{23}$$

We assume the same conditions as those under which Eq. (*1*) in ensemble average is obtained, i.e. the conditions when the radiation term $\partial/\partial t \sum\sum$ is negligible. Then

$$\nabla \cdot \langle \boldsymbol{T}_f \rangle = \langle \mathbf{f}(A \to A) \rangle\,. \tag{24}$$

From Eqs (*20*) and (*24*), we find that the total pressure tensor $\langle \boldsymbol{T}_A \rangle = \langle \boldsymbol{T}_p \rangle + \langle \boldsymbol{T}_f \rangle$ has the divergence

$$\nabla \cdot \langle \boldsymbol{T}_A \rangle = \langle \varrho \rangle\, \boldsymbol{\Gamma} - \langle \mathbf{f}(\setminus A \to A) \rangle\,, \tag{25}$$

which by its form is identical with Eq. (*1*).

We know that the radiation term appearing in the classical three-dimensional divergence (*22*) of $\boldsymbol{T}_f$ disappears if we consider the three-dimensional part of the four-dimensional divergence $\nabla \cdot \boldsymbol{T}_f$, the average of which is just given by Eq. (*24*). A relativistic formalism may hardly be considered here, because definition (*4*) is instantaneous and purely spatial: it remains meaningful for each observer but defines a parameter $\langle G \rangle$ depending on the observer. On the contrary, spatial averages are compatible with a quantum formalism[8], in which the discrete sums above are replaced by strictly instantaneous densities.

Spatial Average at the Face of a Body

Understanding the behaviour of spatial averages[9] at the boundary of a body is important for a correct derivation of the macroscopic electromagnetic equations. The same holds for Eq. (*25*).

Eq. (*25*) may be applied to a body consisting of a small number of particles or a great number of particles. Let us consider the case where A is the body bounded by a closed surface, in a continuous medium N. Let us assume, for instance, that the medium N undergoes a slow deformation ($\boldsymbol{V}$ small, $\boldsymbol{\Gamma}$ negligible), and let the field of displacements be denoted by $\delta\boldsymbol{\xi}$.

The work of the macroscopic distribution of external force acting upon A is obtained from Eq. (25) as

$$\delta W = -\int \mathrm{d}v(\nabla \cdot \langle \mathbf{T}_A \rangle) \cdot \boldsymbol{\delta\xi} \,. \tag{26}$$

In principle, the integration is to be performed over the whole space, but, in practice, it is reduced to a neighborhood of the volume occupied by A. More precisely, it is reduced as noted above with R being the physical radius of the support of the filter function a. Let A_1 and A_2 be surfaces parallel to the surface of A at the distance R, inside and outside A respectively. Outside A_2, $\nabla \cdot \langle \mathbf{T}_A \rangle$ is zero, as it follows from Eq. (25), and δW expressed as*

$$\delta W = -\int_{A_1} - \int_{A_2 \setminus A_1} - \int_{\setminus A_2} \tag{27}$$

reduces therefore to two first terms. Owing to the discontinuity of the body A, $\langle \mathbf{T}_A \rangle$ varies very rapidly between A_1 and A_2 in the direction $\mathbf{n}$ perpendicular to the surface of A. By integrating $\nabla \cdot \langle \mathbf{T}_A \rangle$ first in this direction, i.e. on a segment of the length of $2R$ (on which $\boldsymbol{\delta\xi}$ is almost uniform), one obtains the jump $[\langle \mathbf{T}_A \rangle] = \langle \mathbf{T}_A \rangle (A_2) - \langle \mathbf{T}_A \rangle (A_1)$ of $\langle \mathbf{T}_A \rangle$. Eq. (26) therefore becomes

$$\delta W = -\int_{A_1} \mathrm{d}v(\nabla \cdot \langle \mathbf{T}_A \rangle) \cdot \delta\xi - \iint_A \mathrm{d}s\, \mathbf{n} \cdot [\langle \mathbf{T}_A \rangle] \cdot \boldsymbol{\delta\xi} \,. \tag{28}$$

In common cases (nonpolarized media), $[\langle \mathbf{T}_A \rangle]$ equals to the pressure tensor $-\langle \mathbf{T}_A \rangle$ calculated in A at a distance R from the face, that is calculated on A_1 as in the first term of Eq. (28). This is because $\langle \mathbf{T}_A \rangle (A_2)$ is zero: the pressure of particles $\langle (\mathbf{T}_A)_p \rangle (A_2)$ is zero by definition, since no particle of A exists within a radius R around any observation point $\mathbf{x}$ located on A_2, and the pressure of the field $\langle (\mathbf{T}_A)_f \rangle (\mathbf{x})$ is zero because A creates a zero electromagnetic field outside A_2.

On the contrary, we have $\langle (\mathbf{T}_A)_f \rangle (A_2) \neq \mathbf{O}$ and $[\langle \mathbf{T}_A \rangle] = \langle (\mathbf{T}_A)_f \rangle (A_2) - \langle \mathbf{T}_A \rangle (A_1)$ for a continuous medium which is electrically or magnetically polarized[10]. It may be noted that the calculation of $\langle (\mathbf{T}_A)_f \rangle$ is simpler on A_2 because of the distance R which separates the observation point $\mathbf{x}$ from any particle of A. For example, the electric term of $\mathbf{T}_f$ can be expressed[11] as a function solely of the macroscopic field $\langle \mathbf{E}_A \rangle = \mathbf{E}_A * a$ because in $\langle \mathbf{T}_f \rangle$ the microscopic field $\mathbf{E}_i(\mathbf{x})$ created at point $\mathbf{x}$ by particle i of A is equal to the contribution of i to the macroscopic field $\langle \mathbf{E} \rangle (\mathbf{x})$ observed at $\mathbf{x}$

$$\langle \mathbf{E}_i \rangle (\mathbf{x}) = \int_0^R a(r)\, \mathrm{d}r \iint_{(\mathbf{x},r)} \mathbf{E}_i \, \mathrm{d}s \,. \tag{29}$$

This is a consequence of a mathematical property: the average value on a sphere $(\mathbf{x}, r)$ of the microscopic field $\mathbf{E}_i$ created by particle i equals $\mathbf{E}_i(\mathbf{x})$ if i is outside this sphere.

* The integrals in Eq. (27) are over the volumes defined by surfaces A_1 and A_2, respectively.

In the case of polarized media, Eq. (*25*) or (*28*) can also be applied to a body A which includes, in addition, part of the polarizing external bodies, provided of course that care is taken to define the pressure tensor $\langle \mathbf{T}_A \rangle$ and the external force $\langle \mathbf{f} \rangle = \langle \mathbf{f}(\setminus A \to A) \rangle$ with reference to the *same* body A.

CONCLUSION

The Cauchy equation may be written in spatial average in the same form as in ensemble average (Eq. (*25*) becomes equivalent to Eq. (*1*)). This is a consequence of a great formal parallelism between the derivations of Eq. (*1*) in ensemble and spatial averages. The spatially averaged Eq. (*1*) is therefore valid even in the case where continuous medium N exerts long-range forces on itself. However, in this case, $\mathbf{n} \, . \, \langle \mathbf{T}_N \rangle$ cannot be interpreted as a surface density of contact pressure force, and Eq. (*1*) cannot be treated in the usual way noted in the introduction.

At a point $\mathbf{x}$ inside a medium N in equilibrium ($\mathbf{V} = \mathbf{O}, \boldsymbol{\Gamma} = \mathbf{O}$), Eq. (*25*) can be applied either to medium N, or to the union $M \cup N$ of N and of the external bodies $M \equiv \setminus N$ (the Earth being generally excluded from M as above). The corresponding equations

$$\nabla \, . \, \langle \mathbf{T}_N \rangle (\mathbf{x}) = -\langle \mathbf{f}(M \to N) \rangle - \langle \varrho \rangle \, \mathbf{g} \tag{30}$$

$$\nabla \, . \, \langle \mathbf{T}_{M \cup N} \rangle (\mathbf{x}) = -\langle \varrho \rangle \, \mathbf{g} \tag{31}$$

differ from each other if M exerts long-range forces on N.

Furthermore, the interpretation of $\nabla \, . \, \langle \mathbf{T}_N \rangle$ as the resultant force $\mathbf{F}(N \setminus B \to B)$ exerted on a unit elementary volume B of the medium by the rest $N \setminus B$ of the medium is wrong, except for spherical B: in fact, if the medium is in equilibrium, we do not have only $\boldsymbol{\Gamma} = \mathbf{O}$, but in most common cases also

$$\sum_{i \in N} m_i \gamma_i \delta_i * a = \mathbf{O} \, . \tag{32}$$

Therefore, we have

$$\nabla \, . \, \langle \mathbf{T}_N \rangle = \nabla \, . \, \langle (\mathbf{T}_f)_N \rangle = \langle \mathbf{f}(N \to N) \rangle \, . \tag{33}$$

If B is chosen to be spherical and of radius R around the point $\mathbf{x}$, and the filter function a is chosen to be uniform in the ball $(\mathbf{O}, R)$ (of value $(\frac{4}{3}\pi R^3)^{-1}$), then the macroscopic force distribution $\langle \mathbf{f}(N \to N) \rangle (\mathbf{x})$ coincides with the resultant force $\mathbf{F}(N \setminus B \to B)$ divided by the volume of B. In such a case we have

$$\nabla \, . \, \langle \mathbf{T}_N \rangle (\mathbf{x}) = (\tfrac{4}{3}\pi R^3)^{-1} \, \mathbf{F}(N \setminus B \to B) \, , \tag{34}$$

and the same reasoning applied to $M \cup N$ shows that

$$\nabla . \langle \mathbf{T}_{M \cup N} \rangle (\mathbf{x}) = (\tfrac{4}{3}\pi R^3)^{-1} \mathbf{F}(M \cup N \setminus B \rightarrow B) . \qquad (35)$$

The interpretation (*34*) of $\nabla . \langle \mathbf{T}_N \rangle$ does not apply to nonspherical B and it must not be confused with the relation

$$\int_B \mathrm{d}v \, \nabla . (\mathbf{T}_f)_N = \mathbf{F}(N \setminus B \rightarrow B) \qquad (36)$$

(obtained by integrating Eq. (*22*)), in which a microscopic field pressure tensor is involved; Eq. (*36*) applies to any B.

In ref.[12] we have assumed a special form and orientation for volume B: Eq. (*25*) can then be applied to B itself as in the preceding section, by choosing dimensions of B which are great with respect to R but small compared to the dimensions of the systems. Eq. (*25*) is then written as

$$\nabla . \langle \mathbf{T}_B \rangle = -\langle \varrho \rangle \, \mathbf{g} - \langle \mathbf{f}(M \cup N \setminus B \rightarrow B) \rangle , \qquad (37)$$

where the force distribution $\nabla . \langle \mathbf{T}_B \rangle = \langle \mathbf{f}(B \rightarrow B) \rangle$ is generally not zero. Eq. (*37*) must be distinguished from Eq. (*34*) or (*35*) in which the external force is of positive sign.

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